

Social Indices, Ordinal Scaling and The Rank Dependent Expectation Utility

Indici Sociali, Scale Ordinali ed Utilità Attesa Dipendente dal Rango

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Riassunto: per valutare la performance di agenti (scenari) sociali, si interpreta un tipo di indice sociale sul modello normativo "rank dependent expected utility" (RDEU), formalmente sviluppato nella teoria delle decisioni in contesti economici. Si delinea quindi un approccio alla costruzione di indici che siano suscettibili di essere adattati, non solo a dati statistici (in un'ottica "descrittiva"), ma anche a criteri normativi, tipicamente conformi a relazioni di "preferenza sociale" non necessariamente riducibili ad ordinamenti stocastici

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1. Introduction

The aim of this work is to delineate connections among the "rank dependent expected utility" (RDEU) theory⁽¹⁾, ordinal scaling, social indices and statistical modeling. The specific purpose is to outline process which elicits a cardinal value index for evaluating social agents. A type of problem is the following: evaluate social agents (e.g. formative courses) $A := \{a_1, a_2, \dots, a_p\}$. Outcomes (e.g. results of tests) of micro-agents (students) are revealed over an ordinal scale Y which has L performance levels (e.g. $Y \in \{"Poor", "BelowAverage", "Average", "VeryGood", "Excellent"\}$).

Of course, a performance index is pertinent and useful to policy-maker(PM), whenever it serves to detect, to distinguish and order, among agents (scenarios) according to the specific social criterion the PM would adopt. Thus, there is the necessity to elicit indexes, at least coherent with usual stochastic ordering, which order also scenarios conform to specific "social preference" requirements, for instance due to social risk aversion.

In section 2, we remark connections between RDEU and ordinal scaling. Then, in section 3 we interpret a type of social evaluation index over the RDEU framework. Finally in section 4, we delineate construction of a social index, accounted for by some social criterion which is specified through a weighting function.

2. Rank dependent expected utility and ordinal scaling

Let $Y \in \{"1", "2", \dots, "L"\}$ denote the outcome, for a generic micro-agent, on the L levels of a scale of performance which labels degree of excellence. Let D denote the type of social domain of interest, $\delta := (p_1, \dots, p_L)$ the relative frequency distribution,

⁽¹⁾ Within "the decision theory", literature about RDEU (see Chateauneuf *et al.*, 2004 for references) has been developed in economical settings. But, RDEU seems almost unknown in statistical literature

$\sum_{r=1}^L p_r = 1$. For the sake of convenience, we could interpret $p_r := \Pr\{Y = "r"\}$. Associated to the distribution δ , the cumulative distribution is denoted by $F_Y[\delta]$ so that $F_Y[\delta](r) := p_1 + \dots + p_r$ is the cumulative frequency up to include the level r -th.

The basic (likert-like) quantifying system $\mathcal{U}_0$ identifies the label for l -th level with the cardinal value l itself, so that $\mathcal{U}_0 : \{"1", "2", \dots, "L"\} \rightarrow \mathbb{R}$ and $\mathcal{U}_0("l") = l, l := 1, \dots, L$. Thus, a strictly increasing and continuous function $u : \mathbb{R} \rightarrow \mathbb{R}$ identifies⁽²⁾ a specific scale quantification $\mathcal{U}(\cdot)$, up to a group of affine transformations.

Over the scale of Y on the social domain D , a "social scenario" is described by the discrete distribution $(u(1), p_1; \dots; u(L), p_L)$. Therefore, each social agent may be characterized by a distribution δ , given a quantification function $u(\cdot)$. Recalling and re-interpreting now terms⁽³⁾ within the decision theory setup, the choice model RDEU provides representation theorems which yield (Chateauneuf *et al.*, 2004, pp. 559) the following discrete Choquet's integral (the Quiggin-Yaari's functional):

$$V[\delta; u(\cdot), \varphi(\cdot)] := u(1) + \sum_{r=1}^{L-1} (u(r+1) - u(r)) \varphi(1 - F_Y[\delta](r)) \quad (1)$$

Here, a strictly increasing and continuous function $\varphi : p \in [0, 1] \mapsto \varphi(p) \in [0, 1]$ is considered which weights probabilities (for intuitive reasons, see Weber *et al.*, 1997, pp. 42) subjected to constraints $\varphi(0) = 0$ and $\varphi(1) = 1$. $V[\delta; u(\cdot), \varphi(\cdot)]$ is cardinal in the sense that it is "interval invariant" up to a group of positive affine⁽⁴⁾ transformation of scale quantifying function $u(\cdot)$. In comparing social scenarios, (1) automatically satisfies formal properties of coherence⁽⁵⁾. It being modulated through both the functions $u(\cdot)$ and $\varphi(\cdot)$, RDEU representation (1) perhaps is the ideal archetype for a social index, which would adhere (besides a "descriptive" perspective) also to "normative" requirements base on "social preference".

3. A type of social index

Let values $\omega_r := u(r+1) - u(r) \geq 0$ ($r := 1, \dots, L-1$) have been preliminarily assigned. Suppose that ω_r describes the (pro-capita up to an arbitrary constant of scale) "added social utility", which is associated to the performance of a generic (in the social domain type D) micro-agent who passes from level r -th to $(r+1)$ -th of Y . Then, over domain D , an overall ordinal social evaluation index is provided (recalled in D'Epifanio, 2005) by: $S[\delta; \omega] := \sum_{r=1}^{L-1} c(\omega) \cdot \omega_r \cdot (1 - F_Y[\delta](r))$, where $c(\omega) = 1 / \sum_{r=1}^{L-1} \omega_r$. Typically, usual quantification considers $\tilde{\omega}_r := \omega_r = 1$ ($r := 1, \dots, L-1$), which yields the ordinal index provided in Bernardi *et al* (2004). Of course⁽⁶⁾, a generic ordinal index may be incoherent with a given "social preference" structure.

⁽²⁾ by using function composition: $\mathcal{U}(y) := u(\mathcal{U}_0(y))$, $y \in \{"1", "2", \dots, "L"\}$

⁽³⁾ the "performance levels" as "consequences", the quantification function $u(\cdot)$ as utility, the "social scenario" as "lottery"

⁽⁴⁾ that is: for arbitrary $a > 0, b$ and $\varphi(\cdot)$, $V[\delta; a \cdot u(\cdot) + b, \varphi(\cdot)] = a \cdot V[\delta; u(\cdot), \varphi(\cdot)] + b$

⁽⁵⁾ Let $u_r := u(r)$. For arbitrary $u(\cdot)$, such that $\forall r := 1, \dots, L-1, u_{r+1} \geq u_r$, and weighting function $\varphi(\cdot)$, $V(\cdot)$ preserves the first order stochastic ordering. Furthermore when $u''(\cdot) < 0$ so that $\forall r := 1, \dots, L-2, u_{r+1} - u_r \geq u_{r+2} - u_{r+1}$, it automatically preserves also the II order stochastic ordering

⁽⁶⁾ Consider the scenarios $\delta_A := (0.30, 0.10, 0, 0.20, 0.40)$ and $\delta_B := c(0, 0.20, 0.60, 0.20, 0)$. Then, $S[\delta_A; \tilde{\omega}] = 0.575$, $S[\delta_B; \tilde{\omega}] = 0.5$. But, scenario δ_B might be preferred to δ_A , whenever some type of

Sometimes $\omega := (\omega_1, \dots, \omega_{L-1})$ is a parameter profile which may be modulated (D’Epifanio, 2005) to conform index to requirements. But, when ω is assigned⁽⁷⁾ to conform index may be problematic. To tackle this problem, we consider the index:

$$S^*[\delta; \omega, \varphi(\cdot)] := \sum_{r=1}^{L-1} c(\omega) \cdot \omega_r \cdot \varphi(1 - F_Y[\delta](r)), \quad (2)$$

which may be modulated through both ”scaling parameters” ω and (increasing continuous) weighting function $\varphi(\cdot)$. Here $c(\omega)$ is a constant for standardization over interval $[0, 1]$. Since (2) is modeled⁽⁸⁾ on functional (1), from RDEU it automatically inherits properties. It is cardinal, standardized invariant for arbitrary expansions/contractions of the ”scaling parameters” ω .

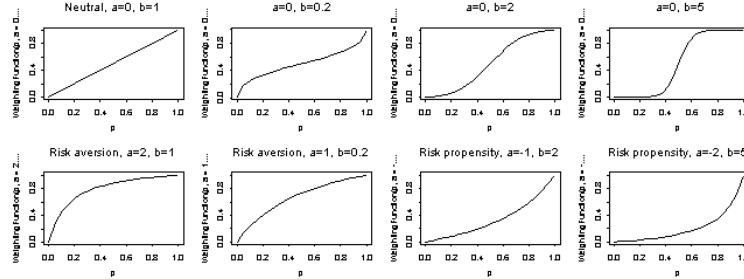
4. About index calibrations

Unfortunately RDEU representations (1-2) provide theoretical results which are difficult to operative in practice. Nevertheless, they provide some guideline.

For the sake of brevity, we assume here that ”incremental values” ω_r^* have been preliminarily⁽⁹⁾ assigned.

Since the function $\varphi(p)$ is difficult to be elicited in concrete problem⁽¹⁰⁾, we would use a specific class of shapes which may be eventually level-parametrized. We suggest the following weighting function: $\varphi(p; a, b) := \frac{e^{a-b \log \frac{1-p}{p}}}{1+e^{a-b \log \frac{1-p}{p}}}$. It is very flexible, regulated⁽¹¹⁾ by parameters $b > 0$ and a (see figure 1).

Figure 1: Weighting function, effects of parameters.



Assume $Pr\{Y = "r"\} > 0$ and let $a := (a_1, \dots, a_{L-1})$, $b := (b_1, \dots, b_{L-1})$.

aversion to social risk is assumed. Therefore, a generic ordinal index may be misleading to PM in ordering scenarios

⁽⁷⁾ approaches may be very different: (a) in decision theory (e.g. in the Neumann and Morgenstern utility theory), or (b) in statistical setting

⁽⁸⁾ letting in (1): $u(1) := 0$, $\omega_r := (u(r+1) - u(r)) \geq 0$, $r := 1, \dots, L-1$,

⁽⁹⁾ For instance, in a statistical setting, ω_r^* might be determined using ordinal models, over a reference population $\mathcal{P}^*$, conform to some ”worthiness” criterion

⁽¹⁰⁾ probability weighting function is the subject of intense investigation. Parametric functions are presented, for example, in Neilson *et al.* (2002); but, their application domain seems to much restrictive

⁽¹¹⁾ By assuming proper conventions about limit passing when $p := 0$ or $p := 1$, for arbitrary $a, b > 0$, we have $\varphi(0^+; a, b) = 0$, $\varphi(1^-; a, b) = 1$; in particular, $\varphi(p; 0, 1) = p$ is the identity.

Sequentially dichotomizing scale of Y , we propose the following index:

$$S^{**}[\delta; \omega^*, a, b] := \sum_{r=1}^{L-1} c(\omega^*, a, b) \cdot \omega_r^* \cdot \frac{e^{a_r - b_r \log \frac{F_Y[\delta](r)}{1 - F_Y[\delta](r)}}}{1 + e^{a_r - b_r \log \frac{F_Y[\delta](r)}{1 - F_Y[\delta](r)}}}, \quad (3)$$

where, a_r, b_r ($b_r > 0$) are level specific parameters. Of course, when $(\forall r) a_r = a_0, b_r = b_0$ (a_0, b_0 are constants) S^{**} conforms to a specific instance of (2); in particular, when, $\forall r, a_r = 0$ and $b_r = 1$, S^{**} coincides with index S (section 3). For a scenarios δ , we could interpret $Z_r[\delta] := -\log \frac{1 - F_Y[\delta](r)}{F_Y[\delta](r)}$ as the score, at level r , of the risk that a generic micro-agent (in social domain D) is unable to pass over level r . The social preference criteria, which the PM adopts, may be modeled by positive affine transformations of the "risk score". Values $b_r > 0$ amplifies/reduces "sensitivity to risk", a_r regulates "propensity" (whenever $a_r < 0$) or "risk aversion" (whenever $a_r > 0$).

Simple algebra directly shows⁽¹²⁾ that $S^{**}[\delta; \omega^*, a, b]$ preserves first order stochastic dominance, whenever $b_r > 0 \forall r := 1, \dots, L-1$. Of course, S^{**} is cardinal invariant for arbitrary expansion/contraction of scaling parameters ω .

In calibrating index, a direct approach before assigns parameters profile which governs weighting, by interpreting implicit preference behavior of PM. The index S^{**} then induces a "weak ordering" (social preferences) among scenarios. Interactive simulations may be useful to check when PM requirements are actually fulfilled, at least on a basic set of reference scenarios. Instead, an indirect approach specifies before a relation of social preferences over a basic finite set of "reference scenarios". Then, heuristic or non-linear programming approaches may be used to calibrate automatically weighting parameters until the assigned requirements are fulfilled.

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⁽¹²⁾ Let consider scenarios δ_A e δ_B . If δ_A is dominated by δ_B then $\forall r, F_Y[\delta_A](r) \geq F_Y[\delta_B](r)$, $\exists r F_Y[\delta_A](r) > F_Y[\delta_B](r)$. Thus $\forall r, a_r + b_r \log \frac{1 - F_Y[\delta_A](r)}{F_Y[\delta_A](r)} \leq a_r + b_r \log \frac{1 - F_Y[\delta_B](r)}{F_Y[\delta_B](r)}$, $\exists r$ for which inequality is strict, provided $b_r > 0$. Therefore, since logistic is strictly increasing, we have $S^{**}[\delta_A; \omega^*, a, b] < S^{**}[\delta_B; \omega^*, a, b]$